Study on the Motion Analysis and Influencing Factors of Simple Pendulum and Double Pendulum

Xuanjin Chen

Guangwai Foreign Language School is Attached to Guangzhou, Guangdong, China

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Abstract: The large swing angle simple pendulum experiment caused systematic errors in periodic measurement due to the neglect of air damping. Based on the motion equation of a single pendulum with weak damping and large swing angles, the analytical formula of the cumulative "period" changing with the pendulum is deduced, and the basic law of the decrease of the "period" with the pendulum is analyzed. Combined with the analysis of experimental data, it is verified that it is used. The system error correction is carried out in the large swing angle simple pendulum experiment, so as to reduce the feasibility of random error through cumulative measurement. A double pendulum formed by two single pendulums connected in series is hung at one end, and two single pendulums with the natural frequency of the double pendulum are hung at the other end. First, push the high-frequency single pendulum as the driving force, and the double pendulum swings in an antisymmetric manner; secondly, push the low-frequency single pendulum gently, and the double pendulum responds in a symmetrical manner; then, push the double pendulum in any way, and the high-frequency single pendulum immediately Responds, resonance occurs, low-frequency pendulum response is very slow, but it can still last for several hours.

1. Introduction

Galileo Galilei (1564-1642), the father of modern science, can apply human reasoning ability to the natural world through the use of mathematics. This is the crystallization of the essence of human reason. Now we know that the laws of nature can be understood through experiments, isolating some unimportant factors, careful observation, and mathematical derivation. This is the worldview of scientists, and it was also the worldview expressed very clearly by Galileo. The success of modern science mainly lies in the use of the quantitative methods revealed by Galileo to describe various phenomena in scientific activities to replace the interpretation of theological philosophy and metaphysics. This is where he surpassed the knowledgeable and talented people of ancient Greece.

Among all the fields of mathematics, differential equations have the closest relationship with nature. The great French mathematician Henri Poincare (1854-1912) once said in the book "The Value of Science":

The science of physics does not only give us (mathematicians) an opportunity to solve problems, but helps us to discover the means of solving them, and it does this in two ways: it leads us to anticipate the solution and suggests suitable lines of argument.

My personal philosophy for the subject of differential equations is just like what British physicist Dirac said, to truly understand physical problems means to see what the answer is without solving equations. "If you believe that differential equations are describing phenomena in nature, then before you solve the equations, it should reveal some secrets to you." The best way to learn mathematics is to start with examples.

Because of my personal preference, I chose the pendulum movement as the object of this article. I heard that Galileo was a boy because the church worship ceremony was very boring and tiresome, so he turned to watch the movement of the church chandelier and found that the pendulum completed a swing. It has nothing to do with its amplitude. This equation is directly a corollary of Newton's second law of motion and has similarities with Hooke's law. This equation appeared even in quantum

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mechanics in the twentieth century, because they are all describing wave phenomena. All differential equations are an approximation, so to truly understand the motion of a simple pendulum depends on an elliptic function. This important science of the nineteenth century did not develop until the soliton theory of integrable systems in the 1970s. We briefly mention this theory in the last section, and leave the readers with more in-depth content for further discussion.

2. The Motion Analysis of the Single Pendulum

2.1 Simple Pendulum Mechanical Analysis

A simple pendulum is an idealized object, assuming a mass point, suspended by a string that cannot be stretched (indicating that the quality of the string can be ignored). Then pull the pendulum to one of the balanced positions and release it. The simple pendulum oscillates left and right due to the influence of gravity. What is the analysis of motion?

From the decomposition of force (that is, the parallelogram rule), the gravity mg of the pendulum can be decomposed into two parts, the normal vector $mg\cos\theta$ and the tangent vector $mg\sin\theta$ (Fig. 1). The normal vector and the string tension balance each other, so the only thing that works the force is $mg\sin\theta$, so the restoring force is

$$F = -mg\sin\theta \qquad (1)$$

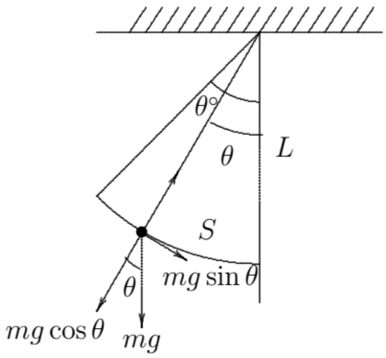

Fig.1 Schematic Diagram of the Force Analysis of a Simple Pendulum

On the other hand, the acceleration of the pendulum can be viewed in this way, because the displacement is the arc length $x = L\theta$, so the acceleration a is equal to

$$a = \frac{d^2x}{dt^2} = \frac{d^2(L\theta)}{dt^2} = L\frac{d^2\theta}{dt^2} \qquad (2)$$

According to Newton's second law of motion ($F = ma$), the equation of motion of a simple pendulum can be derived

$$mL\frac{d^2\theta}{dt^2} = -mg\sin\theta \quad \text{or} \quad \frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0 \qquad (3)$$

It means that the quantity θ under consideration changes according to the law Eq. (3) with time t, which is a second order nonlinear ordinary differential equation. This equation looks simple, but in fact it has profound mathematical connotations. It is worth reminding that eq. (1) tells us: The restoring force F is not proportional to the angle θ of the pendulum movement, but proportional to the $\sin\theta$. Therefore, the pendulum motion is not a simple harmonic motion. However, if the swing angle θ is very small, $\theta \ll 1$, and the pendulum motion is close to a straight line, it can be regarded as simple harmonic motion, ($\sin\theta \ll \theta$).

$$F = -mg\theta = -mg\frac{L\theta}{L} = -\frac{mg}{L}x \qquad (4)$$

Therefore, the swing angle is very small, $0 < \theta \ll 1$, and the restoring force is proportional to the displacement x and the direction is opposite. This is Hooke's law (i.e. simple harmonic motion), and eq. (3) is simplified to

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0 \text{ or } \frac{d^2x}{dt^2} + \frac{g}{L}x = 0 \qquad (5)$$

Therefore, the solution of eq. (5) is the approximate solution of the original pendulum eq. (3). From Taylor's expansion

$$\sin\theta = \frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots \qquad (6)$$

2.2 The Solution of the Simple Pendulum Equation

How to solve the second-order differential eq. (5)? This is a very interesting subject. We will look at this problem from several directions.

2.2.1 Observe and Guess

Shift the Eq. (5) as

$$\frac{d^2x}{dt^2} = -\frac{g}{L}x \qquad (7)$$

This equation tells us that we need to find a function $x(t)$ whose second derivative is equal to the negative value of the original function except for the constant $\frac{g}{L}$, and the pendulum movement has the highest and lowest points, which is equivalent to the function $x(t)$. Upper bound and lower bound. From the knowledge of calculus, it is known that the sine or cosine function has such a property. Therefore, it can be assumed that the solution for eq. (5) is

$$x(t) = A\cos(wt + \varphi) \qquad (8)$$

Because

$$\cos(wt + \varphi) = \cos\varphi\cos wt - \sin\varphi\sin wt = a\cos wt + b\sin wt$$

Therefore, the existence of the constant φ allows eq. (2) to be a combination of sine and cosine functions. (Of course, eq. (2) can also be written as $x(t) = A\sin(wt + \varphi)$!) Differentiate twice and then substitute back to eq. (1):

$$-w^2A\cos(wt + \varphi) = -\frac{g}{L}A\cos(wt + \varphi) \qquad (9)$$

If $w^2 = \frac{g}{L}$, the solution of eq (1) is

$$x(t) = A\cos(wt + \varphi), \quad w^2 = \frac{g}{L} \qquad (10)$$

A and φ have not yet been determined, and they are arbitrary constants, which means that eq. (1) has infinitely many solutions. As for why there are two parameters A and φ, it is completely natural, because eq. (1) is originally a second-order differential equation with two groups of independent solution!

The guess Eq. (2) can be replaced by

$$x(t) = e^{mt} \qquad (11)$$

The reason for this is that any derivative of the exponential function is still an exponential function. Substitute Eq. (12) into Eq. (5)

$$m^2 + \frac{g}{L} = 0 \Rightarrow m = \pm\sqrt{\frac{g}{L}}i \qquad (12)$$

So $x(t) = e^{\pm i\sqrt{g/Lt}}$, take the real and imaginary parts from the Euler formula (because the original equation is a real value!). Therefore $x(t) = \sin\sqrt{g/Lt}$ or $\cos\sqrt{g/Lt}$. As the equation is linear, the general solution is a linear combination of these two independent solutions.

$$x(t) = a\cos\sqrt{\frac{g}{L}}t + b\sin\sqrt{\frac{g}{L}}t \qquad (13)$$

2.2.2 Order Reduction Method

What we care about is the swing of the pendulum, whose rate is the angular velocity, let

$$v = \dot{x} = \frac{dx}{dt} \qquad (14)$$

Then Eq. (1) becomes

$$\dot{v} = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -\frac{g}{L}x \qquad (15)$$

But on the other hand by the chain law

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} \qquad (16)$$

So we convert eq. (1) as a first-order nonlinear differential equation

$$v = \frac{dv}{dx} = -\frac{g}{L}x \text{ or } vdv + \frac{g}{L}xdx = 0 \qquad (17)$$

This is a total differential equation

$$d\left(\frac{1}{2}v^2 + \frac{1}{2}\frac{g}{L}x^2\right) = 0 \qquad (18)$$

Can be integrated as

$$\frac{1}{2}v^2 + \frac{1}{2}\frac{g}{L}x^2 = C \quad (\text{Kinetic + Potential = Constant}) \qquad (19)$$

Where C is a constant, the essence of this equation is the law of conservation of energy or written as

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\frac{g}{L}x^2 = C \qquad (20)$$

We can solve Eq. (21) by the inverse operation of differentiation (integration). For convenience, we set $C = \frac{1}{2}\frac{g}{L}A^2$ then take square root

$$\frac{dx}{dt} = \sqrt{\frac{g}{L}}\sqrt{A^2 - x^2}, \qquad -A \le x \le A \qquad (21)$$

The typical solution of this equation is the method of separation of variables (x goes to left side, t goes to right side)

$$\int \frac{dx}{\sqrt{A^2 - x^2}} = \sqrt{\frac{g}{L}}\int dt \qquad (22)$$

The integral on the left is the inverse trigonometric function

$$\sin^{-1}\left(\frac{x}{A}\right) = \sqrt{\frac{g}{L}}t + C' \text{ or } \cos^{-1}\left(\frac{x}{A}\right) = \sqrt{\frac{g}{L}}t + C' \qquad (23)$$

So

$$x = A\sin\left(\sqrt{\frac{g}{L}}t + C'\right), \quad x = A\cos\left(\sqrt{\frac{g}{L}}t + C'\right) \qquad (24)$$

It is in full agreement with Eq. (14). And from Eq. (22), it can be seen that the maximum value of x is equal to A, so A is the amplitude (amplitude) determined by the initial value, that is, the initial height.

The spirit of the reduction method is to introduce a new variable called velocity $v=\frac{dx}{dt}$. In the theory of differential equations, I call the (x,v) plane the phase space. Therefore, the second-order differential eq. (5) can be transformed into a first-order simultaneous differential equation system

$$\frac{d^2x}{dt^2}+\frac{g}{L}x=0 \Leftrightarrow \begin{cases} \frac{dx}{dt}=v \\ \frac{dv}{dt}=-\frac{g}{L}x \end{cases} \qquad (25)$$

The final division of these two formulas (hiding the variable t) is Eq. (18)

$$\frac{dv}{dx}=-\frac{g}{L}\frac{x}{v} \qquad (26)$$

Therefore, the original differential eq. (5) can be studied on the (x,v) plane. We call the (x,v) plane the phase space. From the perspective of energy conservation, we can understand more clearly why the concept of phase space is so important for the study of differential equations.

2.3 Conservation of Energy

Since eq. (5) is derived from Newton's law, we should naturally think about it from the perspective of mechanics. First introduce a few basic physical quantities

x : Position function

$v=\frac{dx}{dt}$: Velocity

$a=\frac{dv}{dt}=\frac{d^2x}{dt^2}$: Acceleration

$T=\frac{1}{2}v^2=\frac{1}{2}\left(\frac{dx}{dt}\right)^2$: Kinetic energy (mass quality is regarded as 1)

In addition, the relationship between force and potential energy function $F=-\frac{dU}{dx}$, so Eq. (5) can be rewritten as

$$\frac{d^2x}{dt^2}=-\frac{g}{L}x=-\frac{d}{dx}\left(\frac{1}{2}\frac{g}{L}x^2\right)=-\frac{d}{dx}U \qquad (27)$$

where $U=\frac{1}{2}\frac{g}{L}x^2$ is potential energy, while $E=T+U=\frac{1}{2}\left(\frac{dx}{dt}\right)^2+\frac{1}{2}\frac{g}{L}x^2$ is the total energy, we have derived this energy conservation law in Eq. (20)

Theorem: (Conservation of Energy):

$$E(t)=\frac{1}{2}\left(\frac{dx}{dt}\right)^2+\frac{1}{2}\left(\frac{g}{L}\right)x^2=C \text{ (constant)} \qquad (28)$$

Proof: direct differentiation

$$\begin{aligned} \frac{dE}{dt}&=\frac{dx}{dt}\frac{d^2x}{dt^2}+\frac{g}{L}x\frac{dx}{dt} \\ &=\frac{d}{dt}\left(\frac{d^2x}{dt^2}+\frac{g}{L}x\right)=0 \end{aligned} \qquad (29)$$

Special attention is paid to the fact that the entire proof process does not rely on the exact solution of the differential equation. The differential equation itself already tells us the secret. Reverse from the proof process, if you directly multiply $\frac{dx}{dt}$, you can also get the law of conservation of energy, and get the kinetic energy T and the potential energy U.

$$\frac{dx}{dt}\left(\frac{d^2x}{dt^2}+\frac{g}{L}x\right)=\frac{d}{dt}\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2+\frac{1}{2}\frac{g}{L}x^2\right]=\frac{d}{dt}\left(T+U\right) \qquad (30)$$

In the derivation process, it is also clear that the kinetic energy T is derived from acceleration, and the potential energy U is the result of force

$$\begin{array}{ccccc} \frac{d^2x}{dt^2} & + & \frac{g}{L}x & = & 0 \\ \downarrow & & \downarrow & & \\ T & & U & & \end{array}$$

Eq. (28) can be regarded as the Hamilton-Jacobi equation of classical mechanics. In addition to the conservation of apparent energy, since C is an arbitrary constant, it also represents the level curve (Fig. 2), so $v=\frac{dx}{dt}$ is regarded as another Variable, then

$$E\left(t\right)=E\left(x,v\right)=\frac{1}{2}v^2+\frac{1}{2}\frac{g}{L}x^2=C\geq 0 \qquad (31)$$

The figure is an ellipse family in the $\left(x,v\right)$ plane, and if $g=L$, it is a circle.

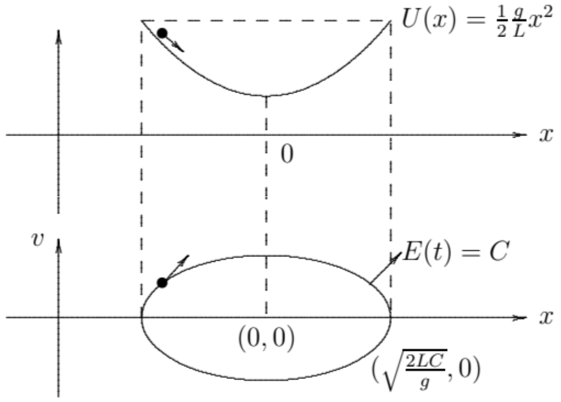

Fig.2 The Level Curves.

3. The Performance of Double Pendulum Frequency with Resonance Phenomenon

3.1 Introduction

Coupled vibrations commonly appear in high school mechanics textbooks, and three springs are connected in series with two vibrators as an example. On the one hand, the series of springs can be presented in the form of Newtonian equations of motion and then drawn in the form of drawing, so that students can easily understand the coupled motion through the senses. On the other hand, the motion of the vibrator makes it easier for students to correspond to the structure of atoms and molecules, and to understand many properties in materials science. However, using spring series vibrators as a demonstration teaching aid for coupled vibration cannot be made in real life. In addition to the influence of gravity, the inevitable torsion movement during the expansion and contraction of the spring is also an important factor.

In Fowles and Cassiday’s Analytical Mechanics, we provide a practical example of making coupled vibration demonstration teaching aids, that is, a single pendulum with the same mass and same length is connected in series to form a coupled pendulum (hereinafter we collectively refer to it as double pendulum), and the double pendulum vibrates It is commonly found in many mechanics textbooks on the description of coupled vibration : two vibrators, two degrees of freedom motion, eigenvector solutions must be symmetrical and antisymmetrical motion; Fowles and Cassiday explained the actual The equation presents the two-dimensional eigenvectors of the vibration of the double pendulum and two natural frequency solutions (also called eigenfrequency), which provide me with the elements for making teaching aids; in addition, in order to present the two natural frequencies of the double pendulum, the rod of the double pendulum is suspended At the same time, two single pendulums with the same natural frequency are hung up, resonating with the double pendulum.

In the history of science, the research on the equations of multi-particle coupled motion began with Newton. However, the biggest contributor was the Bernoulli family who obtained a more general understanding of the Lagrange equation of motion [1]. This is not only the beginning of mathematical physics, but also the coupling Motion also provides a fuller understanding of many aspects, including: atomic and molecular motion, wave science, quantum physics, etc.

3.2 Research Status

Teaching aids similar to the resonant ball structure are often seen in the catalogs or demonstration teaching videos of the demonstration teaching aid vendors. With the length, thickness and support of the rod, there will be different natural frequencies, and almost the same objects have and are similar to nature. The frequency allows teachers to demonstrate the occurrence of resonance. Double pendulums are commonly seen in teaching or demonstrations in shops and online platforms. However, the physical pendulum is the physical pendulum in the video demonstration activity. Instead of connecting two single pendulums in series, the presentation is a large swing and the violent transformation of kinetic energy and potential energy. The process, this is the occurrence of chaos; in academic theories or teaching principle demonstrations, computer animation is often used to illustrate the relationship between position and time in the equation of motion. The design of the double pendulum teaching aid in the report combines the resonance phenomenon with the double pendulum, and returns to the initial assumption of simple harmonic motion: the amplitude is small, so that the double pendulum and the single pendulum with the same natural frequency resonate, specifically understand that the double pendulum has two natural characteristics The frequency, the resonance of different structures, the interesting symmetry and the anti-symmetrical movement make the pendulum present an alternative form of simple beauty.

Therefore, under the assumption of small amplitude, we describe the principles underlying the demonstration from three aspects: simple harmonic motion of a simple pendulum and spring, qualitative and semi-quantitative description of resonance phenomena, and the solution and description of the Lagrange equation of motion for coupled motion. The understanding of pendulum movement began with Galileo’s observation and research on church chandeliers, and the frequency f can be expressed by the following equation.

$$f=\frac{1}{2\pi}\sqrt{g/l} \tag{32}$$

where, □ is the pendulum length, and □ is the acceleration of gravity. Later, we will substitute the two natural frequency solutions obtained by the double pendulum into equation to obtain the pendulum length to be set.

Secondly, for the occurrence of resonance phenomenon, firstly use spring resonance to explain, and then introduce a simple pendulum to explain the difference. For a spring affected by the driving force, the equation of motion can be written according to Newton’s second law of motion.

$$m\ddot{x}+a\dot{x}+kx=F(w) \tag{33}$$

is the mass of the suspension, k is the elastic coefficient, a is the spring damping coefficient, $F(w)$ is the spring driving force, and x is the amount of change. Here, we use Fig. 8 to qualitatively illustrate the occurrence of resonance: the abscissa in the figure is the driving force frequency, the ordinate is the resonance amplitude of the spring, and the curve is the resonance response of the natural frequency to the drive at different frequencies under different damping effects. , Where the Eq. (72) can be obtained when the driving frequency $w_r^2 = w_0^2 - \gamma^2$, where $2\gamma = a / m$, the spring can reach the maximum amplitude. The different curves in the coordinates in Fig. 8 ($Q = w_0^2 / 2\gamma$) indicate that when the damping is smaller, the driving frequency must be closer to the natural frequency to obtain a more ideal resonance phenomenon.

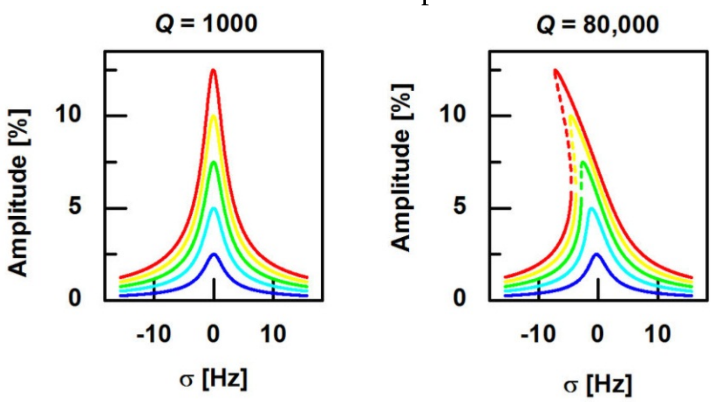

Fig.8 Schematic Diagram of the Relationship between Amplitude and Driving Frequency At Different Q Values

For the vibration of a simple pendulum, the equation of motion (the moment of the pendulum is equal to the moment of gravity) can be written as

$$ml\ddot{\theta} + b\dot{\theta} + mgl\theta = F(w) \quad (34)$$

where b is the damping coefficient when swinging.

In the equipment that uses double pendulum and single pendulum resonance as coupling vibration, we use high-frequency single pendulum and low-frequency single pendulum as driving force respectively to make the double pendulum resonate, and on the other hand, we use double pendulum swing as the driving force. Make the pendulum resonate. Due to the large mass of the pendulum, the air resistance it receives is relatively very small (that is, the damping is small). According to the requirements of the previous resonance performance, the natural frequencies of each pendulum must be very close. This is a requirement for making teaching aids. Eq. (73) It can be simplified as $ml\ddot{\theta} + mgl\theta = F(w)$, eq. (71) is the driving force F equal to 0, the frequency of the pendulum swing.

Finally, the coupled motion of the double pendulum is explained by the Lagrange equation of motion, as shown in Fig. 9. Two double pendulums with the same mass m and the same pendulum length l, when the pendulum angle is very small, its kinetic energy T, potential energy V and Lagrangian can be written as

$$T = \frac{1}{2} ml^2 \left(2\dot{\theta}^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2 \right) \quad (35)$$

$$V \approx mgl\left(2\theta^2 + \phi^2\right) \quad (36)$$

$$L = T - V$$

Fig.9 Double Pendulums

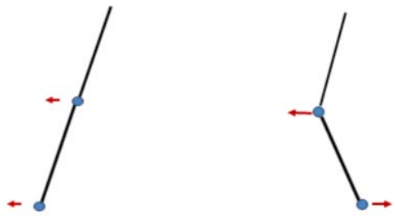

Fig.10 Schematic Diagram of Two Eigenvectors of a Double Pendulum. (a) Symmetry (Left) and (B) Anti-Symmetric (Right)

Where θ and ϕ are the swing angles of the two pendulums of the double pendulum respectively, and the Lagrangian's equation of motion can be written as

$$ml^2\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}\ddot{\theta}\\ \ddot{\phi}\end{bmatrix}+mgl\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\theta\\ \phi\end{bmatrix}=0 \quad (37)$$

Taking the angular velocity function e^{-iwt} into Eq. (76), two eigenfrequency (natural frequency) solutions can be obtained, namely

$$w_s=\left(2-\sqrt{2}\right)^{\frac{1}{2}}\sqrt{g/l} \quad (38)$$

$$w_a=\left(2+\sqrt{2}\right)^{\frac{1}{2}}\sqrt{g/l} \quad (39)$$

w_s is the symmetric solution, w_a is the anti-symmetric solution, and the two eigenvector solutions are.

$$A_s=\begin{bmatrix}1\\ \sqrt{2}\end{bmatrix} \quad (40)$$

$$A_a=\begin{bmatrix}1\\ -\sqrt{2}\end{bmatrix} \quad (41)$$

Where A_s is the symmetric eigenvector, as shown in Fig. 10(a), and A_a is the antisymmetric eigenvector, as shown in Figure 10(b). Eq. (79) and (80) show that when the swing is very small, the value of ϕ is $\sqrt{2}$ times θ, moving in the same direction or in the reverse direction. Their meaning is: when we push the double pendulum arbitrarily, the equation of motion of the double pendulum can be written as

Among them, a b are the amplitudes of two eigenvectors, which are determined by the initial conditions that we push the double pendulum.

In the video, the length of the two pendulums of the double pendulum is 50 cm, the g value is 9.80 m/s^2 into eqs (77) and (78), and then divided by 2π, the symmetric frequency solution is 0.54 Hz, which is antisymmetric. The frequency solution is 1.30 Hz.

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